A note-question on partitions of semigroups

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Abstract. Given a semigroup S and an n-partition $\mathcal{P}$ of S, $n \in \mathbb{N}$, do there exist $A \in \mathcal{P}$ and a subset F of S such that $S = F^{-1}\{x \in S : xA \bigcap A \neq \emptyset\}$ and $|F| \leq n$?

We give an affirmative answer provided that either S is finite or $n = 2$.

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1. Introduction

In 1995, the first author asked the following question [3, Problem 13.44].

Given a group G and an n-partition $\mathcal{P}$, $n \in \mathbb{N}$ of G, do there exist $A \in \mathcal{P}$ and a subset F of G such that $G = FAA^{-1}$ and $|F| \leq n$?

For the current state of this open problem see the survey [1]. We mention only that an answer is positive if either G is amenable (in particular, finite), or $n \leq 3$, or $x^{-1}Ax = A$ for any $A \in \mathcal{P}$ and $x \in G$. If G is an arbitrary group and $\mathcal{P}$ is an n-partition of G then one can choose A, $B \in \mathcal{P}$ and subsets F, H of G such that $G = FAA^{-1}$, $|F| \leq n!$ and $G = HBB^{-1}B$ and $|H| \leq n$.

In this note, we formulate a semigroup version of above question and give positive answer provided that either S is finite or $n = 2$.

For systematic exposition of Ramsey theory of semigroups see [2].

For a semigroup S, $a \in S$, $A \subseteq S$ and $B \subseteq S$, we use the standard notations

$$a^{-1}B = \{x \in S : ax \in B\}, \quad A^{-1}B = \bigcup_{a \in A} a^{-1}B.$$

We set $\Delta(A) = \{x \in S : xA \bigcap A \neq \emptyset\}$ and observe that $\Delta(A) = \{x \in S : x^{-1}A \bigcap A \neq \emptyset\}$ and if S is a group then $\Delta(A) = AA^{-1}$.

We suppose that $S^{-1}A = S$ and define a *covering number*

$$covA = min\{|X| : X \subseteq S, S = X^{-1}A\}.$$

If $S^{-1}A \neq S$ then $covA$ is not defined. Clearly, $covA$ is defined if and only if $Sx \bigcap A \neq \emptyset$ for every $x \in S$.

Now we are ready for promised question.

Given a semigroup S and an n-partition $\mathcal{P}$ of S, does there exist $A \in \mathcal{P}$ such that $cov\Delta(A) \leq n$?

2. Results

Theorem 1. *For a semigroup S and an n-partition $\mathcal{P}$ of S, there exists $A \in \mathcal{P}$ such that $cov\Delta(A) \leq 2^{2^{n-1}-1}$.*

If $n=2$ then $cov\Delta(A)\le 2$. In a personal communication, G. Bergman answered the question positively for $n=3$, and noticed that, we may suppose that a semigroup S is a monoid.

Theorem 2. *For a finite semigroup S and an n-partition $\mathcal{P}$ of S, there exists $A\in\mathcal{P}$ such that $cov\Delta(A)\le n$.*

Theorem 3. *If a subset A of a semigroup S contains either left or right zero then $cov\Delta(A)=1$.*

3. Proofs

Proof of Theorem 1. We adopt arguments from [4, pp. 120-121] proving this theorem for groups.

We define a function $f:\mathbb{N}\times\mathbb{N}\longrightarrow\mathbb{N}$ by

$$f(1,m)=m \;\; and \;\; f(n+1,m)=f(n,m+m^2).$$

By [4, Lemma 12.2], $f(n,m)\le 2^{2^{n-1}-1}m^{2^{n-1}}$.

We use induction on n to prove the following auxiliary statement

($*$) *Let $F, A_1, A_2, \dots, A_n$ be subsets of a semigroup S such that $S=F^{-1}(A_1\bigcup A_2\bigcup\cdots\bigcup A_n)$ and $|F|\le m$. Then there exist $i\in\{1,2,\dots,n\}$ and a subset K of S such that $S=K^{-1}\Delta(A_i)$ and $|K|\le f(n,m)$.*

For $n=1$, we have $S=F^{-1}A_1$. We take an arbitrary $x\in S$ and choose $g\in F$ such that $xA_1\bigcap g^{-1}A_1\ne\emptyset$. Then $A_1\bigcap x^{-1}g^{-1}A_1\ne\emptyset$, $A_1\bigcap(gx)^{-1}A_1\ne\emptyset$ so $gx\in\Delta(A)$, $x\in g^{-1}\Delta(A)$, $x\in F^{-1}\Delta(A)$ and $S=F^{-1}\Delta(A)$.

Let $S=F^{-1}(A_1\bigcup A_2\bigcup\dots\bigcup A_{n+1})$. We consider two cases.

Case 1. $gA_1\subseteq F^{-1}(A_2\bigcup\dots\bigcup A_{n+1})$ for some $g\in S$. Then $A_1\subseteq g^{-1}F^{-1}(A_2\dots A_{n+1})$ and $S=(F^{-1}\bigcup F^{-1}g^{-1}F^{-1})(A_2\bigcup\cdots\bigcup A_{n+1})$. Since $\mid F\bigcup FgF\mid\le m+m^2$, by the inductive hypothesis, there exist $i\in\{2,3,\dots,n+1\}$ and a subset K of S such that

$$S=K^{-1}\Delta(A_i),\quad |K|\le f(n,m+m^2)=f(n+1,m).$$

Case 2. $xA_1\bigcap F^{-1}A_1\ne\emptyset$ for every $x\in S$. Then $A_1\bigcap x^{-1}F^{-1}A_1\ne\emptyset$, $x^{-1}g^{-1}A_1\bigcap A_1\ne\emptyset$ for some $g\in F$, $gx\in\Delta(A_1)$ and $x\in g^{-1}\Delta(A)$, $S=F^{-1}\Delta(A)$. We set $K=F$ and note that $|K|\le m\le f(n+1,m)$.

To conclude the proof, we assume that $S=A_1\bigcup\ldots\bigcup A_n$, take an arbitrary $g\in S$, put $F=\{g\}$, note that $S=F^{-1}(A_1\bigcup\ldots\bigcup A_n)$ and apply ($*$).

Proof of Theorem 2. Let S be a finite semigroup and $S=A_1\bigcup\dots\bigcup A_n$. We take a minimal right ideal R of S, choose $r\in S$ and note that $rS\subseteq R$, $S\subseteq r^{-1}R$, so we may suppose that $S=R$. By [2, Theorem 1.63(g)], S is a direct product of a group G and a right zero semigroup I. We take $a\in I$ and put $H=G\times\{a\}$. For each $i\in\{1,\dots,n\}$, we denote $B_i=A_i\bigcap H$. Since H is a finite group, there are $j\in\{1,\dots,n\}$ and $K\in H$ such that $|K|\le n$ and $H=K^{-1}\Delta_H(B_j)$, where $\Delta_H(B_j)=\{x\in H: xB_j\bigcap B_j\ne\emptyset\}$. We take an arbitrary $(g,b)\in G\times I$ and choose

$z \in K$ such that $z(g,a) \in \{x \in H : xB_j \bigcap B_j \neq \emptyset\}$. Since I is a right zero semigroup, we have $z(g,b)B_j \bigcap B_j \neq \emptyset$. Hence $(g,b) \in z^{-1}\{x \in S : xA_j \bigcap A_j \neq \emptyset\}$ and $S = K^{-1}\Delta(A_j)$.

Proof of Theorem 3. If $a \in A$ is left zero then, for every $x \in S$, we have $S = a^{-1}a = a^{-1}\{x \in A : xA \bigcap A \neq \emptyset\}$ and $S = a^{-1}\Delta(A)$.

If $a \in A$ is right zero then, for every $x \in S$, $a \in xA \bigcap A$ so $xA \bigcap A \neq \emptyset$ and $S = \Delta(A)$ and $S = g^{-1}\Delta(A)$ for each $g \in S$.

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